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# An infection-age-structured modelling for Monkey-Pox disease dynamics incorporating control measures

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**Abstract**

Monkey-pox is known as pathogens affecting livestock animals and humans and belongs to the orthopox virus. The pathogen causes lymph nodes to swell and increasing transmission risk associated with factors involving introduction of virus to the oral mucosa. In this paper, we developed an Age-Structured model for Monkey-pox disease in a population with vital dynamics, incorporating standard incidence rate and vaccination. We showed the existence and uniqueness of the solution of the model. We obtained the Disease-Free Equilibrium state and shown the effective reproduction number of the model. We proved the conditions for Local and Global Stability of the Disease-Free Equilibrium (DFE) State and we found that the disease free equilibrium state is locally asymptotically stable if $G(0) < 1$ $(R_E < 1)$ and Globally Asymptotically Stable (GAS) in $\Omega$ if $R_E \leq 1$ while unstable if $R_E \geq 1$.

***Keywords***: disease-free equilibrium; mathematical modeling; Monkey Pox disease

## 1 Introduction

Monkey pox, Small pox and Cow pox diseases belong to the family of orthopox viruses cause infection in humans and primates [1]. Monkey pox causes lymph nodes to swell up. The symptoms are fever, headache, muscle aches, backache, swollen lymph nodes, a feeling of discomfort and exhaustion. The virus spread to humans from an infected animal (rodents) through direct contact, animal bite and eating infected animal meats without proper done or cook. The virus spread from infected human to human, less infectious than small pox virus, transmitted by direct contact with body fluids of an infected person, contaminated objects and sexual intercourse. The risk factors for transmission include sharing a bed, room, or using the same utensils as an infected patient. Increased

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transmission risk associated with factors involving introduction of virus to the oral mucosa. [2]. Presently, there are no vaccines for monkey pox, however evidence showed that small pox vaccine reduced the risk of monkey pox among previously vaccinated persons in Africa [3]. The incubation period is 7–14 days, the illness lasted for 2 to 4 weeks [1] and the fatality ratio is 1% to 10%. [4]. Therefore, there is non-numerous work on mathematical modeling of monkey pox. This paper reviewed the work of [5]. The organization of paper is as follows, in the next section; the formulation of monkey pox model is presented, Existence and Uniqueness of Solution of the Model, Effective Reproduction Number, Local Stability of the Disease Free Equilibrium (DFE) State, Global Stability of the DFE State are presented and analyzed.

## 2 Model formulation

We formulate a model for the spread of Monkey-pox disease in human and Monkey population. This study reviewed the work of [6] on a model the dynamics of monkey-pox by incorporating the Age-of-Infection to the dynamic of monkey-pox disease. The populations are compartmentalized into epidemiological classes. The total human population is divided into four subgroups that is the Susceptible, $S_h(t)$, the Vaccinated, $V_h(t)$, the Infected, $I_h(t)$, and the Recovery with permanent immunity, $R_h(t)$. The total Monkey population model is divided into the Susceptible, $S_p(t)$, the Vaccinated, $V_p(t)$, the Infected primates, $I_p(t)$, and the recovery with permanent immunity, $R_p(t)$. As shown in the model, people enter Susceptible class through birth and immigration, $(\Pi_h)$, where a proportion of vaccinated human immigrants (f) enter to the vaccinated class and proportion of unvaccinated immigrants (1-f) enter to the susceptible class. We do not consider the immigration of infection person, because we assume that people who are coming from monkey-pox endemic zones have to be vaccinated. The susceptible human individuals vaccinated at the rate $\gamma_h$and loss the vaccination at the rate $\omega_h$. Susceptible human, $S_h$ are exposed to monkey-pox infection at the rate $\alpha_h$ and are infected at the rate $\beta_h$, with natural death $\mu_h$ and die due to the infection at the rate $\delta_h$ and recovery with permanent immunity at a rate $\rho_h$. The Susceptible primates, $S_p$ class is generated from the daily recruitment of individuals through birth and immigration at the $\Pi_p$ and natural death rate $\mu_p$. The susceptible primate individuals vaccinated at the rate $\gamma_p$and loss the vaccination at the rate $\omega_p$ and leave the class to infected class at the rate $\beta_p$. Individual primate infected die due to the infection at the rate $\delta_p$ and recovery with the permanent immunity at the rate $\rho_p$. Base on the assumptions and explanation, the model for transmission dynamics of Monkey-pox infection in human and primates (Monkey) is described by a system of both Ordinary and Partial Differential Equations (ODE/PDE) and nonlinear integro-partial differential equations given below:

$$\frac{dS_h}{dt} = (1-f)\Pi_h + \omega_h V_h - \gamma_h S_h - [\frac{\sigma_p I_p}{N_p} + \frac{\sigma_h I_h}{N_h}]S_h - \mu_h S_h \tag{1}$$

$$\frac{\partial i_h(t,\tau)}{\partial t} + \frac{\partial i_h(t,\tau)}{\partial \tau} + (\mu_h + \delta_h + e_h) i_h(t,\tau) = 0 \tag{2}$$

$$\frac{dV_h}{dt} = f\Pi_h + \gamma_h S_h - \omega_h V_h - \mu_h V_h \tag{3}$$

$$\frac{dR_h}{dt} = e_h I_h - \mu_h R_h \tag{4}$$

$$\frac{dS_p}{dt} = (1-g)\Pi_p + \omega_p V_p - \gamma_p S_p - \frac{\sigma_p I_p}{N_p} S_p - \mu_p S_p \tag{5}$$

$$\frac{\partial i_p(t,\tau)}{\partial t} + \frac{\partial i_p(t,\tau)}{\partial \tau} + (\mu_p + \delta_p + e_p) i_p(t,\tau) = 0 \tag{6}$$

$$\frac{dV_p}{dt} = g\Pi_p + \gamma_p S_p - \omega_p V_p - \mu_p V_p \tag{7}$$

$$\frac{dR_p}{dt} = e_p I_p - \mu_p R_p \tag{8}$$

with boundary conditions

$$B_h = i_h(t,0) = \frac{\sigma_p I_p S_h}{N_p} + \frac{\sigma_h I_h S_h}{N_h} \ \& \ B_p = i_p(t,0) = \frac{\sigma_p I_p S_p}{N_p} \tag{9}$$

and initial condition

$$i_h(0,\tau) = i_h(\tau) = \psi_h \ \& \ i_p(0,\tau) = i_p(\tau) = \psi_p \tag{10}$$

where, $I_h = \int_0^T i_h(t,\tau)d\tau$ and $I_p = \int_0^T i_p(t,\tau)d\tau$ (11)

T is the maximum infection age and if $\tau = T$ the infected person die, then implies
$i_h(t,T) = 0$ and $i_p(t,T) = 0$. (12)

Integration of (2) over $\tau$ give,

$$\frac{dI_h}{dt} = \frac{\sigma_p I_p S_h}{N_p} + \frac{\sigma_h I_h S_h}{N_h} - (\mu_h + \delta_h + e_h) I_h \tag{13}$$

Similarly, integrating (6) over $\tau$

$$\int_0^T \left(\frac{\partial i_p(t,\tau)}{\partial t} + \frac{\partial i_p(t,\tau)}{\partial \tau}\right) d\tau + \int_0^T (\mu_p + \delta_p + e_p) i_p(t,\tau) d\tau = 0 \tag{14}$$

Implies, $\frac{dI_p}{dt} = \frac{\sigma_p I_p S_p}{N_p} - (\mu_p + \delta_p + e_p) I_p$. (15)

# 3 Analysis of the model

## 3.1 Existence and Uniqueness of Solution for the Model

We prove the existence and uniqueness of the solution of the problems (2), (6), (9) and (10). We show that solutions of problems (2), (6), (9) and (10) dependently continuously on the initial age of infection. We define the partial derivative part of (2) and (6) give,

$$\frac{\partial i_h(t,\tau)}{\partial t}+\frac{\partial i_h(t,\tau)}{\partial \tau}=\lim_{h\to 0}\frac{i_h(t+h,\tau+h)-i_h(t,\tau)}{h} \tag{16}$$

Let, $f(h)=i_h(t_0+h,\tau_0+h)$ (17)

The ordinary differential equation

$$\frac{df_1(h)}{dt}+(\mu_h+\delta_h+e_h)f_1(h)=0 \tag{18}$$

Equation (18) has a unique solution as follows

$$f_1(h)=f(0)e^{-\int_0^h(\mu_h+\delta_h+e_h)ds} \tag{19}$$

Substitute (17) into (19), we have

$$i_h(t_0+h,\tau_0+h)=i_h(t_0,\tau_0)e^{-\int_0^h(\mu_h+\delta_h+e_h)ds} \tag{20}$$

This gives the value of $i_h$ at all points on the characteristics passing through the point $i_h(t_0,\tau_0)$

When $i_h(t_0,\tau_0)=(0,t-\tau);h=t$, we have that

$$i_h(t,\tau)=\psi_h(\tau-t)e^{-\int_0^\tau(\mu_h+\delta_h+e_h)ds};t\le\tau \tag{21}$$

Also, when $i_h(t_0,\tau_0)=(t-\tau,0);h=\tau$, gives

$$i_h(t,\tau)=B_h(t-\tau)e^{-\int_0^\tau(\mu_h+\delta_h+e_h)ds};t>\tau \tag{22}$$

Then equations (21) and (22) give the solution of $i_h(t,\tau)$ in the positive quadrant $t\ge 0;\tau\ge 0$. Thus, problem (2), (6), (9), (10) and (11) - (12) has the following unique solution that exists for all time

$$i_h(t,\tau)=\begin{cases}B_h(t-\tau)\pi_h(\tau);t>\tau\\ \psi_h(\tau-t)\dfrac{\pi_h(\tau)}{\pi_h(\tau-t)};t<\tau\end{cases} \tag{23}$$

and

$$i_p(t,\tau)=\begin{cases}B_p(t-\tau)\pi_p(\tau);t>\tau\\ \psi_p(\tau-t)\dfrac{\pi_p(\tau)}{\pi_p(\tau-t)};t<\tau\end{cases} \tag{24}$$

where, $\pi_h(\tau)=e^{-\int_0^\tau(\mu_h+\delta_h+e_h)ds}$ and $\pi_p(\tau)=e^{-\int_0^\tau(\mu_p+\delta_p+e_p)ds}$ (25)

Equation (25) denotes the survival probability i.e the probability for an individual to survive to age $\tau$ ; thus, it must be $\pi(\tau)=0 \ni (\pi_h,\pi_p \in \pi)$ and more over according to [7]. The function

$$K_h(\tau)=\psi_h(\tau)\pi_h(\tau);\tau\in[0,T] \tag{26}$$

Gives an infection function of newly infected individuals, it is related to the parameter:

$$\Gamma=\int_0^T\psi_h(\tau)\pi_h(\tau)d\tau \tag{27}$$

Equation (27) presents the total infection rate which is the newly infected population that an infected individual is expected to infect during the infected period we now consider the expected life of an infected individual given below

$$\overline{\pi}_h=\int_0^T\pi_h(\tau)d\tau \tag{28}$$

Equation (28) gives the mean value of the life of an infected individual; actually (28) is better understood if we note that $(\mu_h+\delta_h+e_h)\pi_h(\tau)d\tau$ is the probability for an infected individual to survive to age $\tau$ and then dies in $[\tau,\tau+d\tau]$; thus

$$\begin{aligned}\overline{\pi}_h&=\int_0^T\tau(\mu_h+\delta_h+e_h)\pi_h(\tau)d\tau=-\int_0^T\tau\frac{d}{d\tau}\pi(\tau)d\tau=-\int_0^T\tau d[\pi(\tau)]\\&=-\tau\pi(\tau)\Big|_0^T+\int_0^T\pi(\tau)d\tau=\int_0^T\pi(\tau)d\tau\end{aligned} \tag{29}$$

We list some of the assumptions that equation (2), (9) and (10) are supposed to fulfil in order to show that (2), (9) and (10) depend continuously on the initial age of infection and are biologically significant or mathematically well posed.

$\psi_h(\cdot)$ is non-negative and belong to $L^T(0,\tau)$ (30)

$(\cdot)$ is non-negative and belong to $L^1{}_{loc}([0,\tau))$ (31)

$$\int_0^\tau(\mu_h+\delta_h(s)+e_h(s))ds=T \tag{32}$$

$$\psi_h \in L^1(0,\tau), \psi_h(\tau) \geq 0 \text{ a.e in } [0,\tau] \tag{33}$$

We rewritten (9) in the form

$$B(t) = B_h + B_p = i_h(t,0) + i_p(t,0) = \frac{\sigma_p I_p S_h}{N_p} + \frac{\sigma_h I_h S_h}{N_h} + \frac{\sigma_p I_p S_p}{N_p}$$

$$B(t) = B_h + B_p = \int_0^T M_1(\tau) i_p(t,\tau) d\tau + \int_0^T M_2(\tau) i_h(t,\tau) d\tau + \int_0^T M_3(\tau) i_p(t,\tau) d\tau \tag{34}$$

where, $M_1 = \frac{\sigma_p S_h}{N_p}; M_2 = \frac{\sigma_h S_h}{N_h}; M_3 = \frac{\sigma_p S_p}{N_p}$ (35)

Equation (35) is also non-negative and belongs to $L^T(0,\tau)$ substitute (23) - (24) into (34), we obtain B(t) as,

$$B(t) = \int_0^T M_1(\tau) B_p(t-\tau) \pi_p(\tau) d\tau + \int_0^T M_1(\tau) \psi_p(\tau - t) \frac{\pi_p(\tau)}{\pi_p(\tau - t)} d\tau + \int_0^T M_2(\tau) B_h(t-\tau) \pi_h(\tau) d\tau$$
$$+ \int_0^T M_2(\tau) \psi_h(\tau - t) \frac{\pi_h(\tau)}{\pi_h(\tau - t)} d\tau + \int_0^T M_3(\tau) B_p(t-\tau) \pi_p(\tau) d\tau + \int_0^T M_3(\tau) \psi_p(\tau - t) \frac{\pi_p(\tau)}{\pi_p(\tau - t)} d\tau \tag{36}$$

Thus, B(t) satisfied the following Volterra Integral equation of the second kind.

$$B(t) = F_1(t) + \int_0^t M_1(\tau) B_p(t-\tau) \pi_p(\tau) d\tau + F_2(t) + \int_0^t M_2(\tau) B_h(t-\tau) \pi_h(\tau) d\tau$$
$$+ F_3(t) + \int_0^t M_3(\tau) B_p(t-\tau) \pi_p(\tau) d\tau \tag{37}$$

with, $F_1(t) = \int_0^T M_1(\tau) \psi_p(\tau - t) \frac{\pi_p(\tau)}{\pi_p(\tau - t)} d\tau = \int_0^T M_1(\tau + t) \psi_p(\tau) \frac{\pi_p(\tau + t)}{\pi_p(\tau)} d\tau$ (38)

$$F_2 = \int_0^T M_2(\tau) \psi_h(\tau - t) \frac{\pi_h(\tau)}{\pi_h(\tau - t)} d\tau = \int_0^T M_2(\tau + t) \psi_h(\tau) \frac{\pi_h(\tau + t)}{\pi_h(\tau)} d\tau \tag{39}$$

$$F_3 = \int_0^T M_3(\tau) \psi_p(\tau - t) \frac{\pi_p(\tau)}{\pi_p(\tau - t)} d\tau = \int_0^T M_3(\tau + t) \psi_p(\tau) \frac{\pi_p(\tau + t)}{\pi_p(\tau)} d\tau \tag{40}$$

and, $K_1(t) = M_1(t)\pi_p(t), K_2(t) = M_2(t)\pi_h(t), K_3(t) = M_3(t)\pi_p(t)$ (41)

implies, $B(t) = F_1(t) + \int_0^T K_1(t-s) B_p(s) ds$

$$+F_2(t)+\int_0^T K_2(t-s)B_h(s)ds+F_3(t)+\int_0^T K_3(t-s)B_p(s)ds \tag{42}$$

When $t\geq 0$ and the functions $M_1, M_2, M_3, \psi_h, \psi_p$ are extended by zero outside the interval [0.T]. equation (37) is known as the renewal equation and also as the Lokta equation, we observed that the Kernal $K_1(t), K_2(t)$ and $K_3(t)$ is an infectious function of newly infected individuals of both human and primate defined in (26). Equation (42) is equivalent to (2), (9), (10) and (11); actually (42) is the main tool to investigate the existence of (2), (9), (10), and (12), the connection being provided by (38), (39), (40) - (41) and (23)-(24) respectively. The following Lemma states some properties on the basis of the assumptions (30) - (33).

**Lemma 1**: Let (38) - (41) be satisfied then,

$K_1(t)$ a.e $K_1(t)=0$ for $t>\tau$ , $K_1\in L^1(\Re_+)\bigcap L^T(\Re_+)$ (43)

$F_1(t)\geq 0, F_2(t)\geq 0, K_1(t)=0$ for $t>\tau$ , $F(F_1,F_2,F_3)\in C(\Re_+)$ (44)

According to [12]

$\psi((\psi_h,\psi_p)\in W^{1,1}(0,\tau)$ and $(\mu_h+\delta_h+e_h)\psi_h(\cdot)\in L^1(0,\tau),\&(\mu_p+\delta_p+e_p)\psi_p(\cdot)\in L^1(0,\tau)$

then $F\in W^{1,T}(\Re_+)$ (45)

**Proof:** Equation (43) and the first part of (44) are obvious. To show that, $F\in C(\Re_+)$ , we take $t\geq 0$ then we have

$$F_1(t)=\int_t^T M_1(\tau)\psi_p((\tau-t)-(\tau-t_0))\frac{\pi_p(\tau)}{\pi_p(\tau-t)}d\tau+\int_t^T M_1(\tau)\psi_p(\tau-t_0)\frac{\pi_p(\tau)}{\pi_p(\tau-t)}d\tau \tag{46}$$

$$F_2(t)=\int_t^T M_2(\tau)\psi_h((\tau-t)-(\tau-t_0))\frac{\pi_h(\tau)}{\pi_h(\tau-t)}d\tau+\int_t^T M_2(\tau)\psi_h(\tau-t_0)\frac{\pi_h(\tau)}{\pi_h(\tau-t)}d\tau \tag{47}$$

$$F_3(t)=\int_t^T M_3(\tau)\psi_p((\tau-t)-(\tau-t_0))\frac{\pi_p(\tau)}{\pi_p(\tau-t)}d\tau+\int_t^T M_3(\tau)\psi_p(\tau-t_0)\frac{\pi_p(\tau)}{\pi_p(\tau-t)}d\tau \tag{48}$$

Since, $\psi_p\in L^1(\Re),\psi_h\in L^1(\Re)$

$$\left|\int_t^T M_1(\tau)\frac{\pi_p(\tau)}{\pi_p(\tau-t)}\psi_p((\tau-t)-(\tau-t_0))d\tau\right|\leq |M_1|_{L^T}\int_0^T\left|\psi_p(\tau-t)-\psi_p(\tau-t_0)\right|d\tau\to 0 \tag{49}$$

As $t\to t_0$

so that, $$\lim_{t\to t_0}F_1(t)=\int_{t_0}^T M_1(\tau)\frac{\pi_h(\tau)}{\pi_h(\tau-t_0)}\psi_h(\tau-t_0)d\tau=F_1(t_0) \tag{50}$$

$$\left|\int_t^T M_2(\tau)\frac{\pi_h(\tau)}{\pi_h(\tau-t)}\psi_h((\tau-t)-(\tau-t_0))d\tau\right|\leq |M_2|_{L^T}\int_0^T\left|\psi_h(\tau-t)-\psi_h(\tau-t_0)\right|d\tau\to 0, t\to t_0 \tag{51}$$

$$\lim_{t\to t_0} F_2(t)=\int_{t_0}^{T} M_2(\tau)\frac{\pi_p(\tau)}{\pi_p(\tau-t_0)}\psi_p(\tau-t_0)d\tau = F_2(t_0) \tag{52}$$

$$\left|\int_{t}^{T} M_3(\tau)\frac{\pi_p(\tau)}{\pi_p(\tau-t)}\psi_p((\tau-t)-(\tau-t_0))d\tau\right| \le |M_3|_{L^T}\int_0^T |\psi_p(\tau-t)-\psi_p(\tau-t_0)|d\tau \to 0, t\to t_0 \tag{53}$$

$$\lim_{t\to t_0} F_3(t)=\int_{t_0}^{T} M_3(\tau)\psi_p(\tau-t_0)\frac{\pi_p(\tau)}{\pi_p(\tau-t_0)}d\tau = F_3(t_0) \tag{54}$$

Therefore, equations (38), (39), (40) implies $F\in W^{1,T}(\Re_+)$. This completes the proof of the lemma 1. In the next theorem, we establish the existence and uniqueness of (2), (9), (10) and (12) by considering (37) - (41). Since the function Fs is continuous, for the purpose of this section it is sufficient to study (37) in the class of continuous functions. Thus, we introduced the following definition of a solution.

**Definition 1**: A Solution to (37) is a function $B(B_h, B_p)\in C(\Re_+,\Re)$ satisfying (37), firstly, we have the following theorem, which is standard in the theory of Volterra Equations. We provide a proof here for the sake of completeness.

**Theorem 1:** Let (30) - (33) be satisfied, then equation (37) - (41) has a unique solution $B(B_h, B_p)\in C(\Re_+,\Re)$ such that $B(t)\ge 0$ for all t in addition if $\psi(\psi_h,\psi_p)$ satisfied (45), then $B\in W^{1,T}_{loc}(\Re_+)$ and

$$\begin{aligned}B^1(t) &= F_1^1(t)+K_1(t)B_p(0)+\int_0^t K_1(t-s)B_p^1(s)ds+F_2^1(t)+K_2(t)B_h(0)\\ &+\int_0^t K_2(t-s)B_h^1(s)ds+F_3^1(t)+K_3(t)B_p(0)+\int_0^t K_3(t-s)B_p^1(s)ds\end{aligned} \tag{55}$$

**Proof:** Firstly, we assume that

$$|K|_{L^1(\Re_+)}=\int_0^T K(s)ds<1 \qquad (K_1+K_2+K_3)\in K \tag{56}$$

The solution of (37) – (42) is obtained using the standard Picard iteration defined, for $t\ge 0$ by

$$B^{k+1}(t)=F_1(t)+\int_0^t K_1(s)B_p^k(t-s)ds+F_2(t)+\int_0^t K_2(s)B_h^k(t-s)ds+F_3(t)+\int_0^t K_3(s)B_p^k(t-s)ds \tag{57}$$

And initialized by $B^0(t)=0$ on IR, therefore,

$$\begin{cases} B^0(t) = F(t) \\ B^{k+1}(t) = F_1(t) + \int_0^t K_1(s) B_p^k(t-s) ds + F_2(t) + \int_0^t K_2(s) B_h^k(t-s) ds + F_3(t) + \int_0^t K_3(s) B_p^k(t-s) ds \end{cases} \quad (58)$$

If we take any $T > 0$, then by (43) - (44) we obtained $B^k \in ([0,T])$ and $B^k(T) \geq 0$, moreover

$$\left|B^{k+1}(t) - B^k(t)\right| \leq \int_0^t K_1(t-s)\left|B_p^k(s) - B_p^{k-1}(s)\right| ds + \int_0^t K_2(t-s)\left|B_h^k(s) - B_h^{k-1}(s)\right| ds$$
$$+ \int_0^t K_3(t-s)\left|B_p^k(s) - B_p^{k-1}(s)\right| ds \quad (59)$$

And,

$$\left|B^{k+1} - B^k\right|_{C([0,T])} \leq \left|K_1\right|_{L^1(\Re_+)}\left|B_p^k - B_p^{k-1}\right|_{C([0,T])} + \left|K_2\right|_{L^1(\Re_+)}\left|B_h^k - B_h^{k-1}\right| + \left|K_3\right|_{L^1(\Re_+)}\left|B_p^k - B_p^{k-1}\right|_{C([0,T])} \quad (60)$$

Thus, by (56) the sequence $B^k(t)$ converges uniformly on $[0,T]$ to a solution of (37) – (42) ∍ $B \in C([0,T])$ and $B(t) \geq 0$. Concerning Uniqueness of this solution of the model, we set B(t) and $\overline{B}$(t) to be the two solutions of (37) - (42), so that

$$\left|B - \overline{B}\right|_{C([0,T])} \leq \left|K\right|_{L^1(\Re_+)}\left|B - \overline{B}\right|_{C([0,T])} \quad (61)$$

This implies that by equation (56), $B(t) = \overline{B}(t)$.

In addition, if $\psi$ satisfies (45) then by theorem J and (58), we have $B^k \in W^{1,T}(\Re_+)$ and setting $V^k(t) = \frac{d}{dt}B^k(t)$ a.e we have $V^k \in L^T(\Re_+)$ and

$$V^{k+1}(t) = F_1^1(t) + K_1(t)V_p(0) + \int_0^t K_1(t-s)V_p^k(s) ds + F_2^1(t) + K_2(t)V_h(0)$$
$$+ \int_0^t K_2(t-s)V_h^k(s) ds + F_3^1(t) + K_3(t)V_p(0) + \int_0^t K_3(t-s)V_p^k(s) ds \quad (62)$$

This yield, $\left|V^{k+1} - V^k\right|_{L^T(\Re_+)} \leq \left|K\right|_{L^1(\Re_+)}\left|V^k - V^{k-1}\right|_{L^T(\Re_+)}$ (63)

Thus, again by (56) the sequence $V^k$ converges in $L^T(\Re_+)$ to $V(t) = \frac{d}{dt}B(t)$ are therefore (55) following from (62). In addition to (57), to make valid argument, we take $\alpha$ ∍

$$\int_0^T e^{-\alpha t} K(t)dt < 1$$

Setting, $\overline{B}_h = e^{-\alpha t} B_h(t), \overline{B}_p = e^{-\alpha t} B_p(t), \overline{F}_1 = e^{-\alpha t} F_1(t), \overline{F}_2 = e^{-\alpha t} F_2(t), \overline{F}_3 = e^{-\alpha t} F_3(t),$
$\overline{K}_h = e^{-\alpha t} K_h(t), \overline{K}_p = e^{-\alpha t} K_p(t)$

in equation (37) - (42) is transformed into an equation below

$$B(t) = \overline{F_1}(t) + \int_0^t \overline{K_1}(t-s)\overline{B_p}(s)ds + \overline{F_2}(t) + \int_0^T \overline{K_2}(t-s)\overline{B_h}(s)ds + \overline{F_3}(t) + \int_0^T \overline{K_3}(t-s)\overline{B_p}(s)ds \tag{64}$$

And since (64) satisfies (56), this is similar with equation (63). The next theorem allows us to state results for problem (2), (9) and (10) through (23) - (24).

**Theorem 2:** let (30) - (33) and (45) be satisfied. And we also assume that

$$\psi_h(0) = \int_0^T M_1(\tau)\psi_h(\tau), \psi_p(0) = \int_0^T M_2(\tau)\psi_p(\tau) \,\&\, \psi_p(0) = \int_0^T M_3(\tau)\psi_p(\tau) \tag{65}$$

And let $i(t,\tau)$ be defined by (23) - (24) where B(t) is the solution of (37) - (42). Then

$$i \in ([0,\tau]\times\Re_+), i(t,\tau) \geq 0, \ (\mu+\delta+e)i(t,\cdot) \in L^1(0,\tau)\forall t > 0 \tag{66}$$

$$\frac{\partial i(t,\tau)}{\partial t} + \frac{\partial i(t,\tau)}{\partial \tau} \text{ exist a.e in } [0,\tau]\times[0,T] \tag{67}$$

And problem (2), (9) and (10) is satisfied. Moreover according to Webb (1985), $i(t,\tau)$ is the only solution in the sense of (57) and (67).

**Proof:**

The proof of (66) - (67) is quite straightforward and follows from the properties of B(t) which is stated in theorem 1. We only note the following inequality concerning the last part of (66).

$$\begin{aligned}
&\int_0^\tau (\mu_h+\delta_h+e_h)i_h(t,\tau)d\tau + \int_0^\tau (\mu_p+\delta_p+e_p)i_p(t,\tau)d\tau = \int_0^{t\wedge\tau} (\mu_h+\delta_h+e_h)B_h(t-\tau)\pi_h(\tau)d\tau \\
&+ \int_{t\wedge\tau}^{\tau} (\mu_h+\delta_h+e_h)\psi_h(\tau-t)\frac{\pi_h(\tau)}{\pi_h(\tau-t)}d\tau + \int_0^{t\wedge\tau}(\mu_p+\delta_p+e_p)B_p(t-\tau)\pi_p(\tau)d\tau \\
&+ \int_{t\wedge\tau}^{\tau} (\mu_p+\delta_p+e_p)\psi_p(\tau-t)\frac{\pi_p(\tau)}{\pi_p(\tau-t)}d\tau
\end{aligned} \tag{68}$$

$$\begin{aligned}
&\leq \underset{S_h\in[0,t]}{Max}\left|B_h(t)\right|\int_0^{t\wedge\tau}(\mu_h+\delta_h+e_h)\pi_h(\tau)d\tau + e^{\int_0^{(t\vee\tau)-t}(\mu_h+\delta_h+e_h)ds}\left|\psi_h\right|_{C(|0,\tau|)}\int_{t\wedge\tau}^{\tau}(\mu_h+\delta_h+e_h)\pi_h(\tau)ds + \\
&\underset{S_p\in[0,t]}{Max}\left|B_p(t)\right|\int_0^{t\wedge\tau}(\mu_p+\delta_p+e_p)\pi_p(\tau)d\tau + e^{\int_0^{(t\vee\tau)-t}(\mu_p+\delta_p+e_p)ds}\left|\psi_p\right|_{C(|0,\tau|)}\int_{t\wedge\tau}^{\tau}(\mu_p+\delta_p+e_p)\pi_p(\tau)ds
\end{aligned} \tag{69}$$

$$\leq \underset{S_h\in[0,t]}{Max}\left|B_h(t)\right| + e^{\int_0^{(t\vee\tau)-t}\mu_h+\delta_h+e_h)ds}\left|\psi_h\right|_{C(|0,\tau|)} + \underset{S_p\in[0,t]}{Max}\left|B_p(t)\right| + e^{\int_0^{(t\vee\tau)-t}\mu_p+\delta_p+e_p)ds}\left|\psi_p\right|_{C(|0,\tau|)} \tag{70}$$

Equation (70) provides the fact that (65) is intended to guarantee the continuity of $i(t,\tau)$ through the line $\tau = t$; so that

$$B_h(0) = \int_0^T M_1(\tau)\psi_h(\tau)d\tau = \psi_h(0) \;\&\; B_p(0) = \int_0^T (M_2 + M_3)(\tau)\psi_h(\tau)d\tau = \psi_p(0) \tag{71}$$

We note that the solution of problem (2), (9) and (10) must be in the form of (23) - (24) with B(t) satisfying (37) - (42) such that equation (23) - (24) and (45) are enough to provide a classical solution in the next following theorem.

**Theorem 3:** Let (30) - (33) be satisfied, then $i(t,\tau)$ defined by (23) - (24) has the following properties;

$$i(t,\cdot) \in C([0,T]; L^1(0,\tau)), i(t,\tau) \geq 0 \ni (i_h + i_p \in i) \text{ a.e in } [0,\tau]\times\Re_+ \tag{72}$$

$$\left|i(t,\cdot)\right|_{L^1} \leq e^{t|M|_{L^T}} \left|\psi\right|_{L^1} \ni (M_1 + M_2 + M_3) \in M, (\psi_h + \psi_p) \in \psi \tag{73}$$

$i(t,\tau)$ is continuous for $\tau < t$ and satisfies (9) for $t > 0$ (74)

$$\lim_{h\to 0}\frac{1}{h}[i(t+h,\tau+h) - i(t,\tau)] = -(\mu+\delta+e)i(t,\tau) \text{ a.e in } [0,\tau]\times\Re_+ \tag{75}$$

**Proof:** let us prove (73) first. From (38) - (42), we have

$$\begin{gathered} F_1(t) \leq \left|M_1\right|_{L^T}\left|\psi_p\right|_{L^1}, K_1(t) \leq \left|M_1\right|_{L^T}; F_2(t) \leq \left|M_2\right|_{L^T}\left|\psi_h\right|_{L^1}, K_2(t) \leq \left|M_2\right|_{L^T}; \\ F_3(t) \leq \left|M_3\right|_{L^T}\left|\psi_p\right|_{L^1}, K_3(t) \leq \left|M_3\right|_{L^T} \end{gathered} \tag{76}$$

Then, from (37) - (42)

$$\begin{aligned} B(t) \leq{}& \left|M_1\right|_{L^T}\left|\psi_p\right|_{L^1} + \left|M_1\right|_{L^T}\int_0^t B_p(s)ds + \left|M_2\right|_{L^T}\left|\psi_h\right|_{L^1} + \left|M_2\right|_{L^T}\int_0^t B_h(s)ds \\ &+ \left|M_3\right|_{L^T}\left|\psi_p\right|_{L^1} + \left|M_3\right|_{L^T}\int_0^t B_h(s)ds \end{aligned} \tag{77}$$

Thus, by classical Gronwall's inequality

$$B(t) \leq \left|M_1\right|_{L^T} e^{t|M_1|_{L^T}}\left|\psi_p\right|_{L^1} + \left|M_2\right|_{L^T} e^{t|M_2|_{L^T}}\left|\psi_h\right|_{L^1} + \left|M_3\right|_{L^T} e^{t|M_3|_{L^T}}\left|\psi_p\right|_{L^1} \tag{78}$$

From this estimate, equation (23) - (24) yields;

$$\begin{aligned} \left|i(t,\cdot)\right|_{L^1} ={}& \int_0^t B_p(t-\tau)\pi_p(\tau)d\tau + \int_0^t \psi_p(\tau)\frac{\pi_p(\tau+t)}{\pi_p(\tau)}d\tau + \int_0^t B_h(t-\tau)\pi_h(\tau)d\tau + \int_0^t \psi_h(\tau)\frac{\pi_h(\tau+t)}{\pi_h(\tau)}d\tau \\ &+ \int_0^t B_p(t-\tau)\pi_p(\tau)d\tau + \int_0^t \psi_p(\tau)\frac{\pi_p(\tau+t)}{\pi_p(\tau)}d\tau \\ \leq{}& (\left|M_1\right|_{L^T}\int_0^t e^{(t-\tau)|M_1|_{L^T}}d\tau + 1)\left|\psi_p\right|_{L^1} + (\left|M_2\right|_{L^T}\int_0^t e^{(t-\tau)|M_2|_{L^T}}d\tau + 1)\left|\psi_h\right|_{L^1} + (\left|M_3\right|_{L^T}\int_0^t e^{(t-\tau)|M_3|_{L^T}}d\tau + 1)\left|\psi_p\right|_{L^1} \\ ={}& e^{t|M_1|_{L^T}}\left|\psi_p\right|_{L^1} + e^{t|M_2|_{L^T}}\left|\psi_h\right|_{L^1} + e^{t|M_3|_{L^T}}\left|\psi_p\right|_{L^1} \end{aligned} \tag{79}$$

Equation (72) follows easily from (75). Now, for a given $\psi \in L^1(0,\tau)$, let $\psi^n$ be a sequence such that $\psi^n$ satisfy (51) and (65)

$$\lim_{n\to T}\left|\psi_h^n - \psi_h\right|_{L^1} = 0 \text{ and } \lim_{n\to T}\left|\psi_p^n - \psi_p\right|_{L^1} = 0 \tag{80}$$

And let $i^n$ be the solution of (2), (9), (10) and (12) corresponding to $\psi^n$. Thus $i^n \in C([0,T];L^1(0,\tau))$ and by (73) and linearity, we have

$$\left|i^n(t,\cdot) - i(t,\cdot)\right|_{L^1} \le e^{t|M|_{L^T}}\left|\psi^n - \psi\right|_{L^1}$$

$$\text{Or}\left|i_h^n(t,\cdot) - i_h(t,\cdot)\right|_{L^1} + \left|i_p^n(t,\cdot) - i_p(t,\cdot)\right|_{L^1} \le e^{t|M_1|_{L^T}}\left|\psi_p^n - \psi_p\right|_{L^1} + e^{t|M_2|_{L^T}}\left|\psi_h^n - \psi_h\right|_{L^1} + e^{t|M_3|_{L^T}}\left|\psi_p^n - \psi_p\right|_{L^1} \tag{81}$$

So that $i$ is the limit of the sequence $i^n$ in the space $C([0,T];L^1(0,\tau))$ i.e (72) is true. Finally, equations (74) - (75) are straightforward and that completes the proof. This shows that even when the initial age of infection $\psi$ is not regular, the solution $i(t,\tau)$ still has some regularity. We also note that the estimate (73) provides continuity of the solution $i$ with respect initial age of infection $\psi$. Hence, model (2), (9) and (10) exist and are unique and well posed mathematically. In the norm of the space $L^1(0,\tau)$: this illustrates the existence and uniqueness of model (2), (9) and (10) which is in agreement with the biological meaning of the infection age density $i(t,\tau)$ for the model.

In the next theorem, we used a similar approach to the work of [8] to prove the existence and uniqueness of solution for the model equations (1)-(8).

**Theorem 4:** The exists of a unique solution of (1) - (8) for $|t-t_0| \le a, |S_h - S_{h0}| \le b$ $, |I_h - I_{h0}| \le c, |V_h - V_{h0}| \le d, |R_h - R_{h0}| \le e, |S_p - S_{p0}| \le f, |I_p - I_{p0}| \le g, |V_p - V_{p0}| \le j, |R_p - R_{p0}| \le l$

**Proof:** To show this, from (1)-(8) yields the following

$$\left|\frac{\partial f_1}{\partial S_h}\right| = \left|+(\gamma_h + \mu_h + \frac{\sigma_p I_p}{N_p} + \frac{\sigma_h I_h}{N_h})\right|, \left|\frac{\partial f_1}{\partial I_h}\right| = \left|\frac{\sigma_h S_h}{N_h}\right|, \left|\frac{\partial f_1}{\partial V_h}\right| = |\omega_h|, \left|\frac{\partial f_1}{\partial I_p}\right| = \left|\frac{\sigma_p S_h}{N_p}\right|, \left|\frac{\partial f_1}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_2}{\partial S_h}\right| = \left|\frac{\sigma_p I_p}{N_p} + \frac{\sigma_h I_h}{N_h}\right|, \left|\frac{\partial f_2}{\partial I_h}\right| = \left|\frac{\sigma_h S_h}{N_h} - (\mu_h + \delta_h + e_h)\right|, \left|\frac{\partial f_2}{\partial I_p}\right| = \left|\frac{\sigma_p S_h}{N_p}\right|, \left|\frac{\partial f_2}{\partial V_p}\right| = |0|, \left|\frac{\partial f_2}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_3}{\partial S_h}\right| = |\gamma_h|, \left|\frac{\partial f_3}{\partial I_h}\right| = |0|, \left|\frac{\partial f_3}{\partial V_h}\right| = |\omega_h + \mu_h|, \left|\frac{\partial f_3}{\partial R_h}\right| = |0|, \left|\frac{\partial f_3}{\partial S_p}\right| = |0|, \left|\frac{\partial f_3}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_4}{\partial S_h}\right| = |0|, \left|\frac{\partial f_4}{\partial I_h}\right| = |e_h|, \left|\frac{\partial f_4}{\partial V_h}\right| = |0|, \left|\frac{\partial f_4}{\partial R_h}\right| = |\mu_h|, \left|\frac{\partial f_4}{\partial S_p}\right| = |0|, \left|\frac{\partial f_4}{\partial I_p}\right| = |0|, \left|\frac{\partial f_4}{\partial V_p}\right| = |0|, \left|\frac{\partial f_4}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_5}{\partial S_h}\right| = |0|, \left|\frac{\partial f_5}{\partial S_p}\right| = \left|+(\gamma_p + \mu + \frac{\sigma_p I_p}{N_p})\right|, \left|\frac{\partial f_5}{\partial I_p}\right| = \left|\frac{\sigma_p S_p}{N_p}\right|, \left|\frac{\partial f_5}{\partial V_p}\right| = |\omega_p|, \left|\frac{\partial f_5}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_6}{\partial S_h}\right| = |0|, \left|\frac{\partial f_6}{\partial R_h}\right| = |0|, \left|\frac{\partial f_6}{\partial S_p}\right| = \left|\frac{\sigma_p I_p}{N_p}\right|, \left|\frac{\partial f_6}{\partial I_p}\right| = \left|\frac{\sigma_p S_p}{N_p} - (\mu_p + \delta_p + e_p)\right|, \left|\frac{\partial f_6}{\partial V_p}\right| = |0|, \left|\frac{\partial f_6}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_7}{\partial S_h}\right| = |0|, \left|\frac{\partial f_7}{\partial I_h}\right| = |0|, \left|\frac{\partial f_7}{\partial V_h}\right| = |0|, \left|\frac{\partial f_7}{\partial R_h}\right| = |0|, \left|\frac{\partial f_7}{\partial S_p}\right| = |\gamma_p|, \left|\frac{\partial f_7}{\partial I_p}\right| = |0|, \left|\frac{\partial f_7}{\partial V_p}\right| = |+\omega_p + \mu_p|, \left|\frac{\partial f_7}{\partial R_p}\right| = |0|$$

$$\left|\frac{\partial f_8}{\partial S_h}\right| = |0|, \left|\frac{\partial f_8}{\partial I_h}\right| = |0|, \left|\frac{\partial f_8}{\partial V_h}\right| = |0|, \left|\frac{\partial f_8}{\partial R_h}\right| = 0, \left|\frac{\partial f_8}{\partial S_p}\right| = |0|, \left|\frac{\partial f_8}{\partial I_p}\right| = |e_p|, \left|\frac{\partial f_8}{\partial V_p}\right| = |0|, \left|\frac{\partial f_8}{\partial R_p}\right| = |\mu_p| \tag{82}$$

This implies that $\left|\frac{\partial f_i}{\partial x_j}\right|, i = 1,2,3,4,5,6,7,8 \Rightarrow j = S_h, I_h, \mathrm{V_h}, R_h, S_p, I_p, \mathrm{V_p}, R_p$ are bounded. Hence, there exists a unique solution of the model (1) - (8).

## 3.2 The existence of equilibrium state

The existence of equilibrium state was done by rewrite equations (1) - (8) as follows;

$$(1-f)\Pi_h + \omega_h V_h - A_1 S_h - \left[\frac{\sigma_p \int_0^T i_p(t,\tau)d\tau}{N_p} + \frac{\sigma_h \int_0^T i_h(t,\tau)d\tau}{N_h}\right] S_h = 0 \tag{83}$$

$$\frac{\partial i_h(t,\tau)}{\partial t} + \frac{\partial i_h(t,\tau)}{\partial \tau} + A_2 i_h(t,\tau) = 0 \tag{84}$$

$$f\Pi_h + \gamma_h S_h - A_3 V_h = 0 \tag{85}$$

$$\int_0^T e_h i_h(t,\tau)d\tau - \mu_h R_h = 0 \tag{86}$$

$$(1-g)\Pi_p + \omega_p V_p - B_1 S_p - \frac{\sigma_p \int_0^T i_p(t,\tau)d\tau}{N_p} S_p = 0 \tag{87}$$

$$\frac{\partial i_p(t,\tau)}{\partial t} + \frac{\partial i_p(t,\tau)}{\partial \tau} + B_2 i_p(t,\tau) = 0 \tag{88}$$

$$g\Pi_p + \gamma_p S_p - B_3 V_p = 0 \tag{89}$$

$$\int_0^T e_p i_p(t,\tau)d\tau - \mu_p R_p = 0 \tag{90}$$

$$B_h = i_h(t,0) = \frac{\sigma_p I_p S_h}{N_p} + \frac{\sigma_h I_h S_h}{N_h} \tag{91}$$

$$B_p = i_p(t,0) = \frac{\sigma_p I_p S_p}{N_p} \tag{92}$$

where,

$A_1 = \gamma_h + \mu_h; A_2 = \mu_h + \delta_h + e_h; A_3 = \omega_h + \mu_h; B_1 = \gamma_p + \mu_p; B_2 = \mu_p + \delta_p + e_p; B_3 = \omega_p + \mu_p$

and $S_h(0) = S_{h0}, I_h(0) = I_{h0}, V_h(0) = V_{h0}, R_h(0) = R_{h0}, S_p(0) = S_{p0}, I_h(0) = I_{p0},$

$$V_p(0) = V_{p0}, R_p(0) = R_{p0}, i_h(0,\tau) = \psi_h, i_p(0,\tau) = \psi_p, N_h{}^0 = S_h^0 + V_h^0, N_p{}^0 = S_p^0 + V_p^0 \tag{93}$$

Solving the system of equations (1) – (8) simultaneously, at the disease free equilibrium state, we have the following points,

$$E^0 = (S_h^0, I_h^0, V_h^0, R_h^0, S_p^0, I_p^0, V_p^0, R_p^0) = \left( \begin{array}{l} \dfrac{(1-f)\Pi_h A_3 + \omega_h f\Pi_h}{(A_1 A_3 - \omega_h \gamma_h)}, 0, \dfrac{f\Pi_h A_1 + \gamma_h (1-f)\Pi_h}{(A_1 A_3 - \omega_h \gamma_h)}, 0, \\ \dfrac{(1-g)\Pi_p B_3 + \omega_p g\Pi_p}{(B_1 B_3 - \omega_p \gamma_p)}, 0, \dfrac{g\Pi_p B_1 + \gamma_p (1-g)\Pi_p}{(B_1 B_3 - \omega_p \gamma_p)}, 0 \end{array} \right) \tag{94}$$

## 3.3 Effective reproduction number

Infection age and age-Structured models, the effective or basic reproduction number are often expressed as the sum of the infectivity of each infected compartment [9, 10]. For a single infected compartment, the basic reproduction number is simply the product of the infection rate and the mean duration of the infection. In view of the above explanation, the effective reproduction number of the model becomes;

$$R_E = R_h + R_p$$

$$R_E = \frac{\sigma_p(B_1B_3-\omega_p\gamma_p)\{(1-f)\Pi_hA_3+\omega_h f\Pi_h\}}{(A_1A_3-\omega_h\gamma_h)\{(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p\}}\int_0^T \pi_p(\tau)d\tau$$
$$+\frac{\sigma_h\{(1-f)\Pi_hA_3+\omega_h f\Pi_h\}}{(1-f)\Pi_hA_3+\omega_h f\Pi_h+f\Pi_hA_1+\gamma_h(1-f)\Pi_h}\int_0^T \pi_h(\tau)d\tau \quad (95)$$
$$+\frac{\sigma_p((1-g)\Pi_pB_3+\omega_p g\Pi_p)}{(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p}\int_0^T \pi_p(\tau)d\tau$$

### 3.4 Local stability of the disease-free equilibrium (DFE) State

**Theorem 5**: The DFE state $E^0$ is locally asymptotically stable if $R_E < 1$ and unstable if $R_E > 1$.
**Proof**: we consider the local stability of DFE state given by (94) and we have the following results,

$$\bar{z}(0) = \frac{\sigma_p(B_1B_3-\omega_p\gamma_p)[(1-f)\Pi_hA_3+\omega_h f\Pi_h]}{[(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p](A_1A_3-\omega_h\gamma_h)}\int_0^T \bar{z}_p(0)e^{-\lambda\tau}\pi_p(\tau)d\tau$$
$$+\frac{\sigma_h[(1-f)\Pi_hA_3+\omega_h f\Pi_h]}{(1-f)\Pi_hA_3+\omega_h f\Pi_h+f\Pi_hA_1+\gamma_h(1-f)\Pi_h}\int_0^T \bar{z}_h(0)e^{-\lambda\tau}\pi_h(\tau)d\tau \quad (96)$$
$$+\frac{\sigma_p[(1-g)\Pi_pB_3+\omega_p g\Pi_p]}{(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p}\int_0^T \bar{z}_p(0)e^{-\lambda\tau}\pi_p(\tau)d\tau$$

From (96), we divide both sides by $\bar{z}(0)$, yields

$$1 = \frac{\sigma_p(B_1B_3-\omega_p\gamma_p)[(1-f)\Pi_hA_3+\omega_h f\Pi_h]Q_1(0)}{[(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p](A_1A_3-\omega_h\gamma_h)}\int_0^T e^{-\lambda\tau}\pi_p(\tau)d\tau$$
$$+\frac{\sigma_h[(1-f)\Pi_hA_3+\omega_h f\Pi_h]Q_2(0)}{(1-f)\Pi_hA_3+\omega_h f\Pi_h+f\Pi_hA_1+\gamma_h(1-f)\Pi_h}\int_0^T e^{-\lambda\tau}\pi_h(\tau)d\tau \quad (97)$$
$$+\frac{\sigma_p[(1-g)\Pi_pB_3+\omega_p g\Pi_p]Q_1(0)}{(1-g)\Pi_pB_3+\omega_p g\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p}\int_0^T e^{-\lambda\tau}\pi_p(\tau)d\tau$$

Where, $Q_1(0)=\bar{z}_p(0)/\bar{z}(0)\ \&\ Q_2(0)=\bar{z}_h(0)/\bar{z}(0)$ (98)

We define a function $G(\lambda)$ to be the right hand side in (97), obviously, $G(\lambda)$ is continuously differentiable function with $\lim_{\lambda\to\infty} G(\lambda)=0$. By direct computation, it can be shown that $G^1(\lambda)<0$ and therefore, $G(\lambda)$ is a decreasing functions. Hence, any real solution of equation (97) is negative if $G(0)<1$ and positive if $G(0)>1$ thus, if $G(0)>1$ , then DFE state is unstable .

Next, we show that equation (97) has no complex solution with non-negative real part if $G(0)<1$ . In fact, we set

$$H_1(\tau)=\frac{\sigma_p(B_1B_3-\omega_p\gamma_p)[(1-f)\Pi_hA_3+\omega_h f\Pi_h]Q_1(0)\pi_p(\tau)}{[(1-g)\Pi_pB_3+\omega_pg\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p](A_1A_3-\omega_h\gamma_h)}$$

$$H_2(\tau)=\frac{\sigma_h[(1-f)\Pi_hA_3+\omega_h f\Pi_h]Q_2(0)\pi_h(\tau)}{(1-f)\Pi_hA_3+\omega_h f\Pi_h+f\Pi_hA_1+\gamma_h(1-f)\Pi_h}$$

$$H_3(\tau)=\frac{\sigma_p[(1-g)\Pi_pB_3+\omega_pg\Pi_p]Q_1(0)\pi_p(\tau)}{(1-g)\Pi_pB_3+\omega_pg\Pi_p+g\Pi_pB_1+\gamma_p(1-g)\Pi_p} \tag{99}$$

$$G(\lambda)=\int_0^T H_1(\tau)e^{-\lambda\tau}d\tau+\int_0^T H_2(\tau)e^{-\lambda\tau}d\tau+\int_0^T H_3(\tau)e^{-\lambda\tau}d\tau \tag{100}$$

Suppose $G(0)<0$. Assume that $\lambda=a_1+ib$ is a complex solution of equation (100) with $a_1\geq 0$ [10]. Then,

$$|G(\lambda)|=\left|\int_0^T H_1(\tau)e^{-\lambda\tau}d\tau+\int_0^T H_2(\tau)e^{-\lambda\tau}d\tau+\int_0^T H_3(\tau)e^{-\lambda\tau}d\tau\right|$$

$$\leq\left|\int_0^T H_1(\tau)e^{-(a_1+ib)\tau}d\tau\right|+\left|\int_0^T H_2(\tau)e^{-(a_1+ib)\tau}d\tau\right|+\left|\int_0^T H_3(\tau)e^{-(a_1+ib)\tau}d\tau\right|$$

$$G(\lambda)\leq\int_0^T\left|e^{-(a_1+ib)\tau}\right|H_1(\tau)d\tau+\int_0^T\left|e^{-(a_1+ib)\tau}\right|H_2(\tau)d\tau+\int_0^T\left|e^{-(a_1+ib)\tau}\right|H_3(\tau)d\tau$$

$$\leq\int_0^T e^{-a_1\tau}H_1(\tau)d\tau+\int_0^T e^{-a_1\tau}H_2(\tau)d\tau+\int_0^T e^{-a_1\tau}H_3(\tau)d\tau \tag{101}$$

$$=G(a_1)\leq G(0)<1 \tag{102}$$

### 3.5 Global stability of the DFE state

**Theorem 6:** The DFE state $E_0$ of the model equation (1) - (5) is Globally Asymptotically Stable (GAS) in $\Omega$ if $R_E\leq 1$ while unstable if $R_E\geq 1$

**Proof:** The approach of [9] and [11-14] are used to construct a suitable lyapunov function,

$$L(t) = (x - x^0 - x^0 \ln\frac{x}{x^0}) + (y - y^0 - y^0 \ln\frac{y}{y^0}) + \int_0^T r(\tau)\frac{i_h(t,\tau)}{\pi_h(\tau)}d\tau$$
$$+ (m - m^0 - m^0 \ln\frac{m}{m^0}) + (j - j^0 - j^0 \ln\frac{j}{j^0}) + \int_0^T q(\tau)\frac{i_p(t,\tau)}{\pi_p(\tau)}d\tau \quad (103)$$

Using (23) - (24) in (103) and solving at DFE, we have the following results,

$$\frac{dL(t)}{dt} = -\frac{A_1}{x}(x - x^0)^2 - \frac{A_3}{y}(y - y^0)^2 - \frac{B_1}{m}(m - m^0)^2 - \frac{B_3}{j}(j - j^0)^2$$
$$- i_h(t,0)(1 - R_h) - i_p(t,0)(1 - R_P) \le 0, if, R_h \le 1, R_p \le 1 \quad (104)$$

The equality $\frac{dL}{dt} = 0$ holds if and only if $x = x^0, y = y^0, m = m^0, j = j^0, i_h(t,0) = 0 \,\&\, i_p(t,0) = 0$. Thus, from the solution (23) – (24) for the model equation (2), (6), (9) along the characteristics lines, we have $i_h(t,0) = 0$ and $i_p(t,0) = 0$ for all $t > \tau$. Hence, $i(t,0) \to 0$ as $t \to \infty$. It can be verified that $\{E^0\}$ is the maximal compact invariant set. Therefore, from the Lassalle invariant principle, we conclude that the disease-free equilibrium $E^0$ is globally asymptomatically stable if $R_E \le 1$

## 4 Conclusion

In this paper, an infection-age-structured model was developed to study the Monkey-pox disease in a population with vital dynamics, incorporating standard incidence rate and vaccination and we analyzed the model by showed the existence and uniqueness of the solution of the model. Meanwhile, we obtained the Disease Free Equilibrium state and shown the effective reproduction number of the model. We showed the conditions for Local and Global Stability of the Disease Free Equilibrium (DFE) State, where we found that the disease free equilibrium state is locally asymptotically stable if $G(0) < 1$ ($\mathrm{R_E} < 1$) and Globally Asymptotically Stable (GAS) if $\mathrm{R_E} \le 1$ while unstable if $\mathrm{R_E} \ge 1$

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